Cambridge International AS & A Level

Mathematics 9709

Paper 1 Pure Mathematics 1

Topic 8-Integration

Question No (2)

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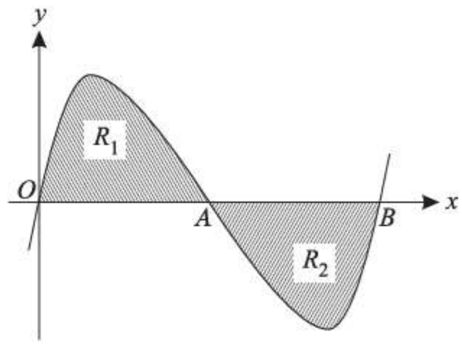

The diagram shows the curve $y = x(x-1)(x-2)$, which crosses the x-axis at the points $O\,(0, 0)$, $A\,(1, 0)$ and $B\,(2, 0)$.

(i) The tangents to the curve at the points A and B meet at the point C. Find the x-coordinate of C.

(ii) Show by integration that the area of the shaded region R_1 is the same as the area of the shaded region R_2.

Solution

On Next page

Given curve

$$y = x(x-1)(x-2) \longrightarrow ①$$

$$y = (x^2-x)(x-2)$$

$$= x^3 - 2x^2 - x^2 + 2x$$

$$y = x^3 - 3x^2 + 2x$$

Differentiate w.r.t x

$$\frac{dy}{dx} = 3x^2 - 3(2x) + 2$$

$$\frac{dy}{dx} = 3x^2 - 6x + 2 \longrightarrow ⓐ$$

Gradient of tangent at $A(1\,0)$

$$\frac{dy}{dx} = 3(1)^2 - 6(1) + 2$$

$$= 3 - 6 + 2$$

$$\frac{dy}{dx} = -1$$

∴ Equation of tangent at A is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - 0 = -1(x-1)$$

$$y = -x + 1 \longrightarrow ②$$

now gradient of tangent at $B(2,0)$

put $B(2,0)$ in ⓐ

$$\frac{dy}{dx} = 3(2)^2 - 6(2) + 2$$

$$= 12 - 12 + 2 = 2$$

$\therefore$ Equation of tangent at $B(2,0)$

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - 0 = 2(x-2)$$

$$y = 2x - 4 \longrightarrow ③$$

solving Equation ② and ③ Simultaneously

$$-x + 1 = 2x - 4$$

$$1 + 4 = 2x + x$$

$$3x = 5$$

$$x = 5/3$$

$\therefore$ x-coordinate of C is $5/3$

(ii) As $y = x(x-1)(x-2)$ given

$$\text{Area of Region } R_1 = \int_0^1 y\,dx$$

$$= \int_0^1 x(x-1)(x-2)\,dx$$

$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$

$$= \left| \frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} \right|_0^1$$

$$= \left[\frac{(1)^4}{4} - \frac{3(1)^3}{3} + \frac{2(1)^2}{2}\right] - \left[\frac{(0)^4}{4} - \frac{3(0)^3}{3} + \frac{2(0)^2}{2}\right]$$

$$= \frac{1}{4} - 1 + 1 = \frac{1}{4}$$

Area of region $R_2 = \int_1^2 y\,dx$

$= \int_1^2 x(x-1)(x-2)\,dx$

$= \int_1^2 (x^3 - 3x^2 + 2x)\,dx$

$= \left| \frac{x^4}{4} - 3\frac{x^3}{3} + 2\frac{x^2}{2} \right|_1^2$

$= \left[\frac{(2)^4}{4} - \frac{3(2)^3}{3} + \frac{2(2)^2}{2}\right] - \left[\frac{(1)^4}{4} - \frac{3(1)^3}{3} + \frac{2(1)^2}{2}\right]$

$= \left[\frac{16}{4} - 8 + 4\right] - \left[\frac{1}{4} - 1 + 1\right]$

$= 4 - 8 + 4 - \frac{1}{4}$

$= -\frac{1}{4}$ (-ve due to below the x-axis)

$\therefore$ area of region $R_2 = \frac{1}{4}$ (absolute value of area)

$\therefore$ both regions R_1 and R_2 have the same area.